平面Green公式的证明

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2017.6.1

1 摘要

本文独立证明了平面Green第一、第三公式.

2 Abstract

This paper proves the first Green's theorem and the third Green's theorem independently.

3 Green第一公式

$$\iint\limits_S [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\mathrm{d}x\mathrm{d}y = -\iint\limits_S u\triangle u\mathrm{d}x\mathrm{d}y + \oint_C u\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}s.$$

证明:

$$\iint\limits_S u\triangle u\mathrm{d}x\mathrm{d}y$$

化为:

$$-\iint\limits_S u(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2})\mathrm{d}x\mathrm{d}y.$$

则:

$$P = u\frac{\partial u}{\partial x}, Q = u\frac{\partial u}{\partial y}.$$

则由Green公式可得:

$$\iint\limits_S u\triangle u\mathrm{d}x\mathrm{d}y = \oint_C P\mathrm{d}x + Q\mathrm{d}y = \oint_C (u\frac{\partial u}{\partial x}cos(\boldsymbol{t}, x) - u\frac{\partial u}{\partial y}cos(\boldsymbol{t}, y))\mathrm{d}s.$$

又:

$$\oint_C u\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}s = \oint_C (u\frac{\partial u}{\partial x}cos(\boldsymbol{n}, x) + u\frac{\partial u}{\partial y}cos(\boldsymbol{n}, y))\mathrm{d}s = \oint_C (u\frac{\partial u}{\partial x}cos(\boldsymbol{t}, y) - u\frac{\partial u}{\partial y}cos(\boldsymbol{t}, x))\mathrm{d}s.$$

其中t为该点的切线向量.
合并上面两式并再次使用Green公式的得:

$$\oint_{c}((u\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial y})cos(\boldsymbol{t},x)+(u\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial y})cos(\boldsymbol{t},y))\mathrm{d}s$$
$$=\iint\limits_{S}[(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2]\mathrm{d}x\mathrm{d}y.$$

证毕.
(Green第二公式教材上已经给出证明，笔者在此就不再赘述.)

4 Green第三公式

若u为有界闭区域S内的调和函数，则:

$$u(x,y)=\frac{1}{2\pi}\oint_{C}(u\frac{\partial \ln r}{\partial \boldsymbol{n}}-\ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s.$$

其中$|r|=\sqrt{(\xi-x)^2+(\eta-y)^2}$为$(x,y)$与$C$上动点$(\xi,\eta)$间的距离.
证明:
因为$\ln r$和u都是调和函数，因此$\triangle\ln r$和$\triangle u$都等于零. 再由Green第二公式可得:

$$\oint_{C+\partial D(CW)}(u\frac{\partial \ln r}{\partial \boldsymbol{n}}-\ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s=0.$$

因此在区域S中的任何子区域D(其中区域D包含点(x,y)):

$$\oint_{C}(u\frac{\partial \ln r}{\partial \boldsymbol{n}}-\ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s=\oint_{\partial D(CCW)}(u\frac{\partial \ln r}{\partial \boldsymbol{n}}-\ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s.$$

假设区域D是以$A(x,y)$为中心，R为半径的一个领域$U(A;R)$，则可得:

$$\iint\limits_{\partial D}\ln r\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}s=\ln R\iint\limits_{\partial D}\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}s.$$

再由Green公式可得上式等于:

$$\ln R\iint\limits_{\partial D}\triangle u\mathrm{d}s=0.$$

且由积分的中值定理可得:

$$\iint\limits_{\partial D}u\frac{\partial \ln r}{\partial \boldsymbol{n}}\mathrm{d}s=\iint\limits_{\partial D}u\frac{\partial \ln r}{\partial r}\mathrm{d}s.$$

$$= \iint\limits_{\partial D} u\frac{1}{R}\mathrm{d}s = \frac{1}{R}\iint\limits_{\partial D} u\mathrm{s} = 2\pi u(\xi^*, \eta^*).$$

其中(ξ^*, η^*)是∂D上一点.
整理得:

$$\oint_C (u\frac{\partial \ln r}{\partial \boldsymbol{n}} - \ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s = \lim_{R\to 0}\oint_{\partial D}(u\frac{\partial \ln r}{\partial \boldsymbol{n}} - \ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s = \lim_{R\to 0} 2\pi u(\xi^*, \eta^*) = 2\pi u(x, y).$$

即:

$$u(x, y) = \frac{1}{2\pi}\oint_C (u\frac{\partial \ln r}{\partial \boldsymbol{n}} - \ln r\frac{\partial u}{\partial \boldsymbol{n}})\mathrm{d}s.$$

证毕.

PS :本文中的公式均独立证明，在今日听了老师讲解的空间格林第三公式的证明之后，我对我的平面格林第三公式的证明进行了细节上的错误的修改.（修改于2017.6.13).